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The Asymptotic Equation and Satellite Conic of the Plane Quartic.

BY TERESA COHEN.

The four tangents at the intersections of a line (ξx) with a quartic curve $(\alpha x)^4$ can theoretically be obtained as follows:

Let y be one of these intersections; then the four tangents are given by eliminating y from

$$\begin{aligned}(\alpha y)^4 &= 0, \text{ the condition } y \text{ be on the quartic,} \\ (\alpha x)(\alpha y)^3 &= 0, \text{ the tangent to the quartic at } y, \\ (\xi y) &= 0, \text{ the condition that } y \text{ be on the line.}\end{aligned}$$

The eliminant is of seventh degree in the coefficients of the quartic, fourth degree in x and twelfth degree in ξ ; all of which may be summed up by saying that it is an $A^7 x^4 \xi^{12}$. For a given ξ it gives the four tangents at the intersections of the line (ξx) with the quartic; hence, it is called the *asymptotic equation* of the quartic. But, equally well, for a given x , it gives the twelve points of contact of tangents from x to the curve.

The four tangents may be thought of as making up a second quartic, and on them and $(\alpha x)^4$ can be built up a whole pencil of quartics, each member of which cuts the quartic twice at the points where (ξx) cuts it. Therefore in the pencil must be a member made up of the line (ξx) counted twice and a conic on the remaining eight points where the four tangents cut the quartic, the so-called *satellite conic*. If $S=0$ be this conic and $T=0$ the four tangents,

$$T + \lambda(\alpha x)^4 = (x\xi)^2 S; \tag{1}$$

or, more symmetrically, since T bears the same relation to $(\alpha\xi)^{12}$, the quartic in lines, as to $(\alpha x)^4$, the quartic in points,

$$T + \lambda(\alpha x)^4(\alpha\xi)^{12} = (x\xi)^2 S. \tag{1'}$$

Since T is an $A^7 x^4 \xi^{12}$, S must be an $A^7 x^2 \xi^{10}$. The object of this paper is to express S , and therefore T , in terms of known comitants of the quartic $(\alpha x)^4$.

The $A^7x^2\xi^{10}$, considered as an equation in x , is the satellite conic of a line ξ ; that is, it is the conic on the eight points where the tangents to the quartic at its intersections with the line ξ again meet the quartic. But considered as an equation in ξ , it is a curve of class 10 on the ten tangents to the quartic from each point of contact of the twelve tangents from a point x . Each double point of the quartic absorbs two of the tangents; therefore, the degree of the conic in ξ decreases by two for each double point, so that the existence of six double points causes the satellite to vanish.

To express this satellite in terms of known forms of the quartic, it is best to take up special cases where the satellite is known. Three such have been found sufficient.

Special Cases.

Case I. $4x_0(x_1^3+x_2^3)=0.$

This represents four lines, three of which are on a point. Owing to the presence of six double points, $S=0$.

Case II. $x_0^4+x_1^4+x_2^4=0.$

To obtain the four tangents it is necessary to eliminate y from

$$y^4+y_1^4+y_2^4=0, \quad x_0y_0^3+x_1y_1^3+x_2y_2^3=0, \quad \xi_0y_0+\xi_1y_1+\xi_2y_2=0.$$

This may be accomplished by means of the eliminant of

$$\alpha_0y_0^4+\alpha_1y_1^4+\alpha_2y_2^4=0, \quad \beta_0y_0^3+\beta_1y_1^3+\beta_2y_2^3=0, \quad y_0+y_1+y_2=0,$$

which is found to be *

$$\begin{aligned} & \sum^3 \alpha_0^3 (\beta_1 - \beta_2)^4 + 3 \sum^6 \alpha_1^2 \alpha_2 \beta_0 \{ \beta_0^3 + 16 \beta_0^2 \beta_2 + 10 \beta_0 \beta_2^2 - 4 \beta_1 (\beta_0 - \beta_2)^2 \} \\ & + 18 \alpha_0 \alpha_1 \alpha_2 \{ 4 (\beta_0 \beta_1 \beta_2 - (\beta_1^2 \beta_2^2)) \}. \end{aligned}$$

Let $y_i = \xi_i y'_i$, $\alpha_i = \frac{x_i}{\xi_i^3}$, $\beta_i = \frac{1}{\xi_i^4}$. Then the second set of equations reduces to the first set and the eliminant, after clearing of fractions, becomes

$$\begin{aligned} & \sum^3 x_0^4 (\xi_1^{12} + 3 \xi_1^8 \xi_2^4 + 3 \xi_1^4 \xi_2^8 + \xi_2^{12}) + \sum^6 4 x_0^3 x_1 \xi_0^3 \xi_1 (-\xi_1^8 + 12 \xi_1^4 \xi_2^4 - 3 \xi_2^8) \\ & + \sum^3 6 x_0^2 x_1^2 \xi_0^2 \xi_1^2 (\xi_0^4 \xi_1^4 + 5 \xi_0^4 \xi_2^4 + 5 \xi_1^4 \xi_2^4 - 3 \xi_2^8) + \sum^3 12 x_0^2 x_1 x_2 \xi_0^2 \xi_1 \xi_2 (2 \xi_0^4 \xi_1^4 + 2 \xi_0^4 \xi_2^4 - \xi_1^8 \\ & + 6 \xi_1^4 \xi_2^4 - \xi_2^8). \end{aligned}$$

This, by equation (1), must be equal to

$$\begin{aligned} & \lambda (x_0^4 + x_1^4 + x_2^4) + (x\xi)^2 [\sum^3 x_0^2 \xi_0^2 \{ A \xi_0^8 + B (\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) + C (\xi_1^8 + \xi_2^8) + D \xi_1^4 \xi_2^4 \} \\ & + \sum^3 2 x_1 x_2 \xi_1 \xi_2 \{ E \xi_0^8 + F (\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) + G (\xi_1^8 + \xi_2^8) + H \xi_1^4 \xi_2^4 \}], \end{aligned}$$

* This is from a remark by Professor Morley.

where the expression in brackets is the satellite conic. Since λ enters into the coefficient of x_i^4 only, A, B , etc., may be obtained from the other coefficients.

$$A = -1, \quad B = -3, \quad C = -3, \quad D = 21, \quad E = -3, \quad F = 3, \quad G = 1, \quad H = 3.$$

$$\begin{aligned} \therefore S = & \sum x_0^2 \xi_0^2 [-\xi_0^8 - 3(\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) - 3(\xi_1^8 + \xi_2^8) + 21\xi_1^4 \xi_2^4] \\ & + \sum x_1 x_2 \xi_1 \xi_2 \cdot 2[-3\xi_0^8 + 3(\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) + (\xi_1^8 + \xi_2^8) + 3\xi_1^4 \xi_2^4]. \end{aligned}$$

Case III. *The Lemniscate:* $6(x_1^2 x_2^2 + x_2^2 x_0^2 + x_0^2 x_1^2) = 0$.*

It is a property of the lemniscate that a tangent at a point y meets the curve again in two points that lie on the conic (y/x) . Now, suppose y eliminated between

$$\frac{y_0}{x_0} + \frac{y_1}{x_1} + \frac{y_2}{x_2} = 0, \quad y_1^2 y_2^2 + y_2^2 y_0^2 + y_0^2 y_1^2 = 0, \quad y_0 \xi_0 + y_1 \xi_1 + y_2 \xi_2 = 0,$$

the operation being a simple one because two of the equations are linear. The eliminant is an octavic, made up of the four conics (y/x) for the four points y in which the line ξ cuts the lemniscate. Since each conic passes through the reference points, which are double points of the lemniscate, the thirty-two intersections of octavic and quartic are made up of the three reference points, each taken eight times, and the eight intersections of the curve with the four tangents at its intersections with the line ξ . Therefore, it must be possible to find a quartic $(A_0 x_1^2 x_2^2) + (B_0 x_0^2 x_1 x_2)$ such that

$$\begin{aligned} & \sum x_1^4 x_2^4 \xi_1^2 \xi_2^2 + \sum x_0^4 x_1^3 x_2 \cdot -2\xi_0^2 \xi_1 \xi_2 + \sum x_0^4 x_1^2 x_2^2 \xi_0^2 (\xi_0^2 + \xi_1^2 + \xi_2^2) \\ & + \sum x_0^3 x_1^3 x_2^2 \cdot 2\xi_0 \xi_1 (-\xi_0^2 - \xi_1^2 + 2\xi_2^2) \\ & + (x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2) [(A_0 x_1^2 x_2^2) + (B_0 x_0^2 x_1 x_2)] = x_0^2 x_1^2 x_2^2 S, \\ & A_0 = -\xi_1^2 \xi_2^2, \quad B_0 = 2\xi_0^2 \xi_1 \xi_2. \end{aligned}$$

Therefore, multiplying by $\xi_0^2 \xi_1^2 \xi_2^2$, the square of the double points,

$$S = \sum x_0^2 \xi_0^6 \xi_1^2 \xi_2^2 + \sum x_1 x_2 \xi_1 \xi_2 \cdot 2[3\xi_0^4 \xi_1^2 \xi_2^2 - (\xi_0^2 \xi_1^4 \xi_2^2 + \xi_0^2 \xi_1^2 \xi_2^4)].$$

The Expression of the Satellite in Terms of Known Comitants of the Quartic.

The satellite, an $A^7 x^2 \xi^{10}$, may be written as

$$S \equiv C_{7,2,10} + (x\xi)C_{7,1,9} + (x\xi)^2 C_{7,0,8}.$$

Therefore it is not necessary to deal with all possible terms at once, since we

* See R. A. Roberts, "Examples on Conics and Higher Plane Curves."

must have that $S - C_{7,2,10} = (x\xi)S'$ (i. e., contains $(x\xi)$ as a factor), and, again, $S' - C_{7,1,9} = (x\xi)S''$.

Let us, then, first consider the possibilities for the $C_{7,2,10}$. If the two simplest contravariants of the quartic be taken as

$$(s\xi)^4 = s \equiv \frac{1}{2} |\alpha\beta\xi|^4 \text{ and } (t\xi)^6 = t \equiv \frac{1}{6} |\alpha\beta\xi|^2 |\beta\gamma\xi|^2 |\gamma\alpha\xi|^2,$$

and if it be understood that

$$s_i s_j s_k = (s\xi)^{4-i} (s\alpha)^i (s'\xi)^{4-j} (s'\alpha)^j (s''\xi)^{4-k} (s''\alpha)^k (\alpha x)^{4-i-j-k}$$

and

$$t_i t_j = (t\xi)^{6-i} (t\alpha)^i (t'\xi)^{6-j} (t'\alpha)^j (\alpha x)^{4-i-j},$$

then

$$C_{7,2,10} = M_{011} s s_1^2 + M_{002} s^2 s_2 + N_{11} t_1^2 + N_{02} t t_2,$$

where the M 's and N 's are coefficients to be determined.

A relation on these coefficients is most easily obtained from Case I, where the satellite vanishes and, therefore, $C_{7,2,10}$ must itself contain $(x\xi)$.

$$\begin{aligned} C_{7,2,10} &= \sum^2 x_0 x_1 [\xi_0^6 \xi_1^4 (8N_{11} + \frac{32}{5} N_{02}) + \xi_0^6 \xi_1 \xi_2^3 (216M_{011} + \frac{32}{5} N_{02})] \\ &\quad + \sum^2 x_1^2 [\xi_0^5 \xi_1^5 (8N_{11} + \frac{48}{5} N_{02}) + \xi_0^5 \xi_1^2 \xi_2^3 (432M_{011} + 576M_{002} + 8N_{11} + \frac{48}{5} N_{02})] \\ &= (x_0 \xi_0 + x_1 \xi_1 + x_2 \xi_2) [x_1 (A \xi_0^5 \xi_1^4 + B \xi_0^5 \xi_1 \xi_2^3) + x_2 (A \xi_0^5 \xi_2^4 + B \xi_0^5 \xi_1 \xi_2^3)] \\ &= \sum^2 (x_0 x_1 \xi_0 + x_1^2 \xi_1) (A \xi_0^5 \xi_1^4 + B \xi_0^5 \xi_1 \xi_2^3) + x_1 x_2 (\xi_0^5 \xi_1^4 \xi_2 + \xi_0^5 \xi_1 \xi_2^4) (A + B). \\ A &= 8N_{11} + \frac{32}{5} N_{02} = 8N_{11} + \frac{48}{5} N_{02} \quad \therefore N_{02} = 0. \\ B &= 216M_{011} + \frac{32}{5} N_{02} = 432M_{011} + 576M_{002} + 8N_{11} + \frac{48}{5} N_{02} \\ &\quad \therefore 216M_{011} + 576M_{002} + 8N_{11} = 0. \\ A + B &= 8N_{11} + 216M_{011} = 0 \quad \therefore M_{002} = 0, \quad M_{011} : N_{11} = 1 : -27 = -6 : 162.* \\ &\therefore S = -6 (s s_1^2 - 27 t_1^2) + (x\xi) S'. \end{aligned}$$

For this case

$$S' = \sum^2 x_1 (-1296 \xi_0^5 \xi_1^4 + 1296 \xi_0^5 \xi_1 \xi_2^3).$$

The correctness of this determination of $C_{7,2,10}$ may be tested on the other cases. For Case II,

$$\begin{aligned} S - C_{7,2,10} &= \lambda [\sum^3 x_0^2 \xi_0^2 \xi_2^2 \{ -\xi_0^8 - 3(\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) - 3(\xi_1^8 + \xi_2^8) + 21 \xi_1^4 \xi_2^4 \} \\ &\quad + \sum^3 x_1 x_2 \xi_1 \xi_2 \{ -6 \xi_0^8 + 6(\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) + 2(\xi_1^8 + \xi_2^8) + 6 \xi_1^4 \xi_2^4 \} \\ &\quad + 6 \sum^3 x_0^2 \xi_0^2 \xi_2^2 \{ \xi_0^8 + (\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) - 3 \xi_1^4 \xi_2^4 \} \\ &= (x\xi) \sum^3 x_0 \xi_0 [a \xi_0^8 + b(\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) + c(\xi_1^8 + \xi_2^8) + d \xi_1^4 \xi_2^4]. \\ \lambda &= 1, a = 5, b = 3, c = -3, d = 3. \\ \therefore S' &= \sum^3 x_0 \xi_0 [5 \xi_0^8 + 3(\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) - 3(\xi_1^8 + \xi_2^8) + 3 \xi_1^4 \xi_2^4]. \end{aligned}$$

* The factor -6 is introduced because it simplifies subsequent work. It is introduced here to avoid confusion.

For Case III,

$$\begin{aligned}
 S - C_{7,2,10} &= \lambda [\sum_0^3 x_0^2 \xi_0^2 \cdot \xi_0^4 \xi_1^2 \xi_2^2 + \sum_1^3 x_1 x_2 \xi_1 \xi_2 \{ 6 \xi_0^4 \xi_1^2 \xi_2^2 - 2 (\xi_0^2 \xi_1^4 \xi_2^2 + \xi_0^2 \xi_1^2 \xi_2^4) \}] \\
 &\quad + 27 \cdot 216 [\sum_0^3 x_0^2 \xi_0^2 \{ 2 \xi_0^4 \xi_1^2 \xi_2^2 - 5 (\xi_0^2 \xi_1^4 \xi_2^2 + \xi_0^2 \xi_1^2 \xi_2^4) + 2 (\xi_1^6 \xi_2^2 + \xi_1^2 \xi_2^6) + 4 \xi_1^4 \xi_2^4 \} \\
 &\quad + \sum_1^3 x_1 x_2 \xi_1 \xi_2 \{ 2 (\xi_0^6 \xi_1^2 + \xi_0^2 \xi_1^6) + 4 (\xi_0^4 \xi_1^4 + \xi_0^2 \xi_1^2 \xi_2^2) - 28 \xi_0^4 \xi_1^2 \xi_2^2 \\
 &\quad + 2 (\xi_0^2 \xi_1^6 + \xi_0^2 \xi_1^2 \xi_2^6) + 6 (\xi_0^2 \xi_1^4 \xi_2^2 + \xi_0^2 \xi_1^2 \xi_2^4) \}] \\
 &= (x\xi) \sum_0^3 x_0 \xi_0 \{ a \xi_0^4 \xi_1^2 \xi_2^2 + b (\xi_0^2 \xi_1^4 \xi_2^2 + \xi_0^2 \xi_1^2 \xi_2^4) + c (\xi_1^6 \xi_2^2 + \xi_1^2 \xi_2^6) + d \xi_1^4 \xi_2^4 \}. \\
 \lambda &= 3 \cdot 27 \cdot 216, \quad a = 5 \cdot 27 \cdot 216, \quad b = -5 \cdot 27 \cdot 216, \quad c = 2 \cdot 27 \cdot 216, \quad d = 4 \cdot 27 \cdot 216. \\
 \therefore S' &= 27 \cdot 216 \sum_0^3 x_0 \xi_0 [5 \xi_0^4 \xi_1^2 \xi_2^2 - 5 (\xi_0^2 \xi_1^4 \xi_2^2 + \xi_0^2 \xi_1^2 \xi_2^4) + 2 (\xi_1^6 \xi_2^2 + \xi_1^2 \xi_2^6) + 4 \xi_1^4 \xi_2^4].
 \end{aligned}$$

Turning now to the $C_{7,1,9}$, we have as possible terms

$$C_{7,1,9} = M_{111} s_1^3 + M_{012} s s_1 s_2 + N_{12} t_1 t_2 + N_{03} t t_3. *$$

Here it is easiest to work first with Case II.

$$\begin{aligned}
 S' - C_{7,1,9} &= \sum_0^3 x_0 \xi_0 [\xi_0^8 (5 - M_{111} - M_{012}) + (\xi_0^4 \xi_1^4 + \xi_0^4 \xi_2^4) (3 - M_{012}) \\
 &\quad - 3 (\xi_1^8 + \xi_2^8) + \xi_1^4 \xi_2^4 (3 - N_{12}^{\frac{12}{45}})].
 \end{aligned}$$

If $(x\xi)$ is to be a factor, the coefficient of $x_i \xi_i$ must be the same for $i=0, 1, 2$; i. e., must be symmetrical in ξ_i . Therefore,

$$\begin{aligned}
 5 - M_{111} - M_{012} &= -3, & 3 - M_{012} &= 3 - N_{12}^{\frac{12}{45}}, \\
 \therefore M_{111} &= 8 - M_{012}, & N_{12} &= 45 M_{012}.
 \end{aligned} \tag{2}$$

In Case I,

$$\begin{aligned}
 S' - C_{7,1,9} &= x_0 \xi_0 (\xi_0^5 \xi_1^3 + \xi_0^5 \xi_2^3) (-27 M_{111} - \frac{8}{5} N_{12} - \frac{8}{5} N_{03}) \\
 &\quad + \sum_1^2 x_1 \xi_1 [\xi_0^5 \xi_1^3 (-1296 - \frac{32}{5} N_{12} - \frac{36}{5} N_{03}) \\
 &\quad + \xi_0^5 \xi_2^3 (1296 - 162 M_{111} - 144 M_{012} - \frac{8}{5} N_{12} - \frac{36}{5} N_{03})], \\
 27 M_{111} + \frac{8}{5} N_{12} + \frac{8}{5} N_{03} &= 1296 + \frac{32}{5} N_{12} + \frac{36}{5} N_{03} = -1296 + 162 M_{111} \\
 &\quad + 144 M_{012} + \frac{8}{5} N_{12} + \frac{36}{5} N_{03}.
 \end{aligned}$$

By use of relations (2), these may be solved.

$$\begin{aligned}
 M_{111} &= \frac{176}{13}, \quad M_{012} = -\frac{72}{13}, \quad N_{12} = -\frac{3240}{13}, \quad N_{03} = \frac{4320}{91}, \\
 \therefore S' &= \frac{8}{13} (22 s_1^3 - 9 s s_1 s_2 - 405 t_1 t_2 + \frac{540}{7} t t_3) + (x\xi) S''.
 \end{aligned}$$

For Case II,

$$S'' = -3 (\xi^8) + \frac{111}{3} (\xi_1^4 \xi_2^4).$$

For Case I,

$$S'' = -\frac{3888}{91} \xi_0^5 (\xi_1^3 + \xi_2^3).$$

* $s^2 s_3$ is a numerical multiple of $(x\xi) s^2 s_4$.

For Case III,

$$\begin{aligned}
 S' - C_{7,1,9} = & \sum_{\xi_0}^3 \xi_0^8 \left(-18M_{012} - \frac{2}{5}N_{12} - \frac{6}{5}N_{03} \right) \\
 & + (\xi_0^6\xi_1^2 + \xi_0^6\xi_2^2) \left(-81M_{111} - 126M_{012} - \frac{2}{5}N_{12} - \frac{3}{5}N_{03} \right) \\
 & + (\xi_0^4\xi_1^4 + \xi_0^4\xi_2^4) \left(-243M_{111} - 270M_{012} - \frac{12}{5}N_{12} + \frac{27}{5}N_{03} \right) \\
 & + \xi_0^4\xi_1^2\xi_2^2 \left(135 \cdot 216 - 486M_{111} - 540M_{012} + \frac{444}{5}N_{12} + \frac{378}{5}N_{03} \right) \\
 & + (\xi_0^2\xi_1^6 + \xi_0^2\xi_2^6) \left(-243M_{111} - 234M_{012} - \frac{26}{5}N_{12} + \frac{39}{5}N_{03} \right) \\
 & + (\xi_0^2\xi_1^4\xi_2^2 + \xi_0^2\xi_1^2\xi_2^4) \left(-135 \cdot 216 - 729M_{111} - 702M_{012} - \frac{978}{5}N_{12} - \frac{693}{5}N_{03} \right) \\
 & + (\xi_1^8 + \xi_2^8) \left(-81M_{111} - 72M_{012} - \frac{14}{5}N_{12} + 3N_{03} \right) \\
 & + (\xi_1^6\xi_2^2 + \xi_1^2\xi_2^6) \left(54 \cdot 216 - 324M_{111} - 288M_{012} + \frac{196}{5}N_{12} + 12N_{03} \right) \\
 & + \xi_1^4\xi_2^4 (108 \cdot 216 - 486M_{111} - 432M_{012} + 84N_{12} + 18N_{03}). \\
 S'' = & \frac{648}{91} [20(\xi^8) - 46(\xi_0^6\xi_1^2) - 132(\xi_1^4\xi_2^4) + 987(\xi_0^4\xi_1^2\xi_2^2)].
 \end{aligned}$$

Finally, for the $C_{7,0,8}$ the possibilities are

$$C_{7,0,8} = M_{004}s^2s_4 + M_{022}ss_2^2 + M_{112}s_1^2s_2 + N_{04}tt_4 + N_{13}t_1t_3 + N_{22}t_2^2.*$$

Instead of dealing with all of these terms at once it is simpler to take

$$C'_{7,0,8} = M_{112}s_1^2s_2 + N_{04}tt_4 + N_{13}t_1t_3 + N_{22}t_2^2.$$

Then, $S'' - C'_{7,0,8} = sS'''$; i. e., contains s as a factor.

For Case I,

$$\begin{aligned}
 S'' - C'_{7,0,8} = & \xi_0^5(\xi_1^3 + \xi_2^3) \left(-\frac{3888}{91} - 36M_{112} - \frac{32}{5}N_{04} - \frac{22}{5}N_{13} - \frac{96}{25}N_{22} \right) = 0, \\
 \therefore & \frac{1944}{91} + 18M_{112} + \frac{16}{5}N_{04} + \frac{11}{5}N_{13} + \frac{48}{25}N_{22} = 0.
 \end{aligned} \tag{3}$$

For Case II,

$$\begin{aligned}
 S'' - C'_{7,0,8} = & \sum_{\xi_0}^3 \xi_0^8 (-3 - M_{112}) + \sum_{\xi_0}^3 \xi_0^4 \xi_2^4 \left(\frac{111}{13} - \frac{1}{225}N_{22} \right) = (\xi^4) \cdot A(\xi^4) \\
 \therefore & A = -3 - M_{112} = \frac{111}{26} - \frac{1}{450}N_{22} \text{ or } -\frac{189}{13} - 2M_{112} + \frac{1}{225}N_{22} = 0.
 \end{aligned} \tag{4}$$

For Case III,

$$\begin{aligned}
 S'' - C'_{7,0,8} = & \sum_{\xi_0}^3 \xi_0^8 \left(\frac{648}{91} \cdot 20 - 18M_{112} + \frac{4}{5}N_{04} - \frac{6}{5}N_{13} - \frac{22}{25}N_{22} \right) \\
 & + \sum_{\xi_0}^6 \xi_0^6 \xi_1^2 \left(-\frac{648}{91} \cdot 46 - 180M_{112} + \frac{16}{5}N_{04} + \frac{18}{5}N_{13} + \frac{32}{25}N_{22} \right) \\
 & + \sum_{\xi_0}^3 \xi_0^4 \xi_1^4 \left(-\frac{648}{91} \cdot 132 - 324M_{112} + \frac{24}{5}N_{04} + \frac{48}{5}N_{13} + \frac{108}{25}N_{22} \right) \\
 & + \sum_{\xi_0}^3 \xi_0^4 \xi_1^2 \xi_2^2 \left(\frac{648}{91} \cdot 987 - 756M_{112} - \frac{168}{5}N_{04} - \frac{294}{5}N_{13} - \frac{2592}{25}N_{22} \right) \\
 = & 3[(\xi^4) + 2(\xi_1^2\xi_2^2)][A(\xi^4) + B(\xi_1^2\xi_2^2)] \\
 = & 3[A(\xi^8) + (B + 2A)(\xi_0^6\xi_1^2) + (2A + 2B)(\xi_1^4\xi_2^4) \\
 & + (2A + 5B)(\xi_0^4\xi_1^2\xi_2^2)].
 \end{aligned}$$

* ss_1s_3 is a numerical multiple of s^2s_4 .

From this comes the single relation

$$-\frac{27}{91} \cdot 1377 + \frac{9}{5}N_{04} + \frac{18}{5}N_{13} + \frac{122}{25}N_{22} = 0. \quad (5)$$

(3), (4), (5) give only three equations to solve for four unknowns. They may be solved for three in terms of the fourth, say M_{112} .

$$N_{04} = 5\left(\frac{9}{91} \cdot 3085 + \frac{920}{21}M_{112}\right), \quad N_{13} = 5\left(\frac{9}{91} \cdot -10280 - \frac{3022}{21}M_{112}\right), \\ N_{22} = 25\left(\frac{9}{91} \cdot 1323 + \frac{378}{21}M_{112}\right).$$

Using these relations to obtain the coefficients of S''' in terms of M_{112} , it remains only to equate S''' to $M_{004}ss_4 + M_{022}s_2^2$.

For Case II,

$$S''' - M_{004}ss_4 - M_{022}s_2^2 = (\xi^4) (-3 - M_{112} - 3M_{004} - M_{022}) = 0. \\ \therefore 3 + 3M_{004} + M_{022} + M_{112} = 0. \quad (6)$$

For Case III,

$$S''' - M_{004}ss_4 - M_{022}s_2^2 = \sum_3 \xi_0^4 \left(\frac{3}{13} \cdot 6622 + \frac{1874}{9}M_{112} - 54M_{004} - 14M_{022} \right) \\ + \sum_3 \xi_1^2 \xi_2^2 \left(-\frac{3}{13} \cdot 27052 - \frac{8228}{9}M_{112} - 108M_{004} - 52M_{022} \right). \\ \therefore \frac{3}{13} \cdot 6622 + \frac{1874}{9}M_{112} - 54M_{004} - 14M_{022} = 0. \quad (7)$$

$$-\frac{3}{13} \cdot 27052 - \frac{8228}{9}M_{112} - 108M_{004} - 54M_{022} = 0. \quad (8)$$

These two relations, together with (6), can be solved for the three unknowns.

$$M_{004} = -\frac{49}{13}, \quad M_{022} = \frac{9}{26} \cdot 45, \quad M_{112} = -\frac{9}{26} \cdot 21, \quad N_{04} = -\frac{9}{91} \cdot 675, \quad N_{13} = \frac{9}{91} \cdot 1485, \quad N_{22} = 0.$$

Therefore,

$$S = -6(ss_1^2 - 27t_1^2) + (x\xi) \cdot \frac{8}{13}(22s_1^3 - 9ss_1s_2 - 405t_1t_2 + \frac{540}{7}tt_3) \\ + (x\xi)^2 \cdot \frac{9}{13} \left(-\frac{49}{9}s^2s_4 + \frac{45}{2}ss_2^2 - \frac{21}{2}s_1^2s_2 - \frac{675}{7}tt_4 + \frac{1485}{7}t_1t_3 \right).$$

The Satellite for the General Quartic.

This expression for the satellite conic may be verified by use of the general quartic

$$ax_0^4 + bx_1^4 + cx_2^4 + 4a_1x_0^3x_1 + 4a_2x_0^3x_2 + 4b_0x_0x_1^3 + 4c_0x_0x_2^3 + 4b_2x_1^3x_2 + 4c_1x_1x_2^3 \\ + 6fx_1^2x_2^2 + 6gx_2^2x_0^2 + 6hx_0^2x_1^2 + 12lx_0^2x_1x_2 + 12mx_0x_1^2x_2 + 12nx_0x_1x_2^2 = 0,$$

where the conic can be found for a special reference triangle and line ξ . Let $(x\xi) \equiv x_0$, and let two of the tangents at its intersections with the quartic be $x_1 = 0$ and $x_2 = 0$. Then, in the equation of the quartic,

$$b = c = b_0 = c_0 = 0.$$

Beside the two reference points $x_0=0$ cuts the quartic in two points given by

$$4b_2x_1^2+6fx_1x_2+4c_1x_2^2=0. \quad (9)$$

Let these two points be $(0, A, 1)$ and $(0, B, 1)$. The tangent at the first of these points is

$$x_0(3mA^2+3nA)+x_1(3b_2A^2+3fA+c_1)+x_2(b_2A^3+3fA^2+3cA)=0,$$

or, making use of the fact that A satisfies (9) and dividing by A ,

$$x_0(3mA+3n)+x_1(2b_2A+\frac{3}{2}f)+x_2(\frac{3}{2}fA+2c_1)=0.$$

Similarly, the tangent at $(0, B, 1)$ is

$$x_0(3mB+3n)+x_1(2b_2B+\frac{3}{2}f)+x_2(\frac{3}{2}fB+2c_1)=0.$$

Multiplying these two together and making use of the symmetric functions of the roots of a quadratic, we have the equation of the two tangents in terms of the coefficients of the quartic. Then by (1),

$$\begin{aligned} 2x_1x_2[x_0^2(72b_2n^2+72c_1m^2-108fmn)+x_0x_1(96b_2c_1m-54f^2m)+x_0x_2(96b_2c_1n-54f^2n) \\ +x_1^2(32b_2^2c_1-18b_2f^2)+x_1x_2(48b_2c_1f-27f^3)+x_2^2(32b_2c_1^2-18c_1f^2)] \\ +\lambda(ax_0^4+4a_1x_0^3x_1+4a_2x_0^2x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2 \\ +12mx_0x_1^2x_2+12nx_0x_1x_2^2+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3)=x_0^3S. \\ \lambda=(-16b_2c_1+9f^2). \end{aligned}$$

$$\begin{aligned} S=x_0^2(-16ab_2c_1+9af^2)+2x_0x_1(-32a_1b_2c_1+18a_1f^2)+2x_0x_2(-32a_2b_2c_1+18a_2f^2) \\ +x_1^2(-96b_2c_1h+54f^2h)+2x_1x_2(-96b_2c_1l+72b_2n^2+72c_1m^2 \\ +54f^2l-108fmn)+x_2^2(-96b_2c_1g+54fg^2). \end{aligned}$$

The formula to be tested is

$$\begin{aligned} S=M_{011}ss_1^2+N_{11}t_1^2+(x\xi)(M_{111}s_1^3+M_{012}ss_1s_2+N_{12}t_1t_2+N_{03}tt_3) \\ +(x\xi)^2(M_{004}s^2s_4+M_{022}ss_2^2+M_{112}s_1^2s_2+N_{04}tt_4+N_{13}t_1t_3), \end{aligned}$$

where

$$\begin{aligned} M_{011}=-6, \quad N_{11}=162, \quad M_{111}=\frac{176}{13}, \quad M_{012}=-\frac{72}{13}, \quad N_{12}=-\frac{3240}{13}, \quad N_{03}=\frac{4320}{91}, \\ M_{004}=-\frac{49}{13}, \quad M_{022}=\frac{9}{26} \cdot 45, \quad M_{112}=-\frac{9}{26} \cdot 21, \quad N_{04}=-\frac{9}{91} \cdot 675, \quad N_{13}=\frac{9}{91} \cdot 1485. \end{aligned}$$

In this case it becomes

$$\begin{aligned} S=x_0^2[ab_2^3c_1^3(-64M_{011}-64M_{111}-64M_{012}-192M_{004}-64M_{022}-64M_{112}=64) \\ +ab_2^2c_1^2f^2(144M_{011}+4N_{11}+144M_{111}+144M_{012}+4N_{12}+4N_{03}+432M_{004} \\ +144M_{022}+144M_{112}+\frac{24}{5}N_{04}+4N_{13}=-36) \\ +ab_2c_1f^4(-108M_{011}-4N_{11}-108M_{111}-108M_{012}-4N_{12}-4N_{03}-324M_{004} \\ -108M_{022}-108M_{112}-\frac{24}{5}N_{04}-4N_{13}=0) \\ +af^6(27M_{011}+N_{11}+27M_{111}+27M_{012}+N_{12}+N_{03}-81M_{004}+27M_{002} \\ +27M_{112}+\frac{6}{5}N_{04}+N_{13}=0) \end{aligned}$$

$$\begin{aligned}
 & + (a_1 b_2^2 c_1^3 m + a_2 b_2^2 c_1^3 n) (96M_{011} + 144M_{111} + 144M_{012} + 576M_{004} + 192M_{022} \\
 & \qquad \qquad \qquad + 192M_{112} \equiv 0) \\
 & + (a_1 b_2^2 c_1^2 f n + a_2 b_2^2 c_1^2 f m) (-96M_{011} - \frac{8}{5}N_{11} - 144M_{111} - 144M_{012} - 4N_{12} \\
 & \qquad \qquad \qquad - \frac{22}{5}N_{03} - 576M_{004} - 192M_{022} - 192M_{112} - \frac{32}{5}N_{04} - \frac{16}{3}N_{13} \equiv 0) \\
 & + (b_2^3 c_1^2 g l + b_2^3 c_1^2 h l) (-64M_{012} - 576M_{004} - 192M_{022} - 64M_{112} \equiv 0) \\
 & + (b_2^3 c_1 f g^2 + b_2^3 c_1 f h^2) (-\frac{4}{5}N_{12} - \frac{12}{5}N_{03} - 4N_{04} - \frac{12}{5}N_{13} \equiv 0) \\
 & + b_2^2 c_1^2 f g h (32M_{012} + \frac{8}{15}N_{12} + \frac{8}{5}N_{03} + 288M_{004} + 64M_{022} + 32M_{112} - \frac{24}{5}N_{04} + \frac{8}{5}N_{13} \equiv 0) \\
 & + (a_1 b_2 c_1^2 f^2 m + a_2 b_2 c_1^2 f^2 n) (-144M_{011} - 4N_{11} - 216M_{111} - 216M_{012} - 6N_{12} \\
 & \qquad \qquad \qquad - \frac{33}{5}N_{03} - 864M_{004} - 288M_{022} - 288M_{112} - \frac{48}{5}N_{04} - 8N_{13} \equiv 0) \\
 & + b_2^2 c_1^2 f l^2 (64M_{012} + \frac{16}{15}N_{12} + \frac{16}{5}N_{03} + 576M_{004} + 224M_{022} + 64M_{112} + \frac{64}{5}N_{04} + \frac{16}{5}N_{13} \equiv 0) \\
 & + (b_2^3 c_1 g n^2 + b_2 c_1^3 h m^2) (-36M_{011} - 108M_{111} - 48M_{012} + \frac{4}{15}N_{12} - 16M_{022} \\
 & \qquad \qquad \qquad - 150M_{112} + \frac{7}{5}N_{13} \equiv 0) \\
 & + (b_2^2 c_1^2 g m^2 + b_2^2 c_1^2 h n^2) (\frac{4}{9}N_{11} + 56M_{012} + \frac{4}{5}N_{12} + 576M_{004} + 192M_{022} + 62M_{112} \\
 & \qquad \qquad \qquad + \frac{7}{15}N_{13} \equiv 0) \\
 & + b_2^2 c_1^2 l m n (-72M_{011} + \frac{8}{9}N_{11} - 216M_{111} - 128M_{012} + \frac{16}{15}N_{12} - 576M_{004} \\
 & \qquad \qquad \qquad - 224M_{022} - 368M_{112} + \frac{56}{15}N_{13} \equiv 0) \\
 & + (a_1 b_2 c_1 f^3 n + a_2 b_2 c_1 f^3 m) (144M_{011} + \frac{16}{3}N_{11} + 216M_{111} + 216M_{012} + 8N_{12} \\
 & \qquad \qquad \qquad + \frac{44}{5}N_{03} + 864M_{004} + 288M_{022} + 288M_{112} + \frac{64}{5}N_{04} + \frac{32}{3}N_{13} \equiv 0) \\
 & + (b_2^2 c_1 f^2 g l + b_2 c_1^2 f^2 h l) (96M_{012} + \frac{8}{5}N_{12} + \frac{24}{5}N_{03} + 864M_{004} + 288M_{022} + 96M_{112} \\
 & \qquad \qquad \qquad + 8N_{04} + \frac{24}{5}N_{13} \equiv 0) \\
 & + (b_2^3 c_1 f g m n + b_2 c_1^2 f h m n) (72M_{011} + \frac{4}{3}N_{11} + 216M_{111} + 84M_{012} + 4N_{12} \\
 & \qquad \qquad \qquad + \frac{27}{5}N_{03} + 32M_{022} + 294M_{112} + 16N_{04} + \frac{41}{5}N_{13} \equiv 0) \\
 & + (b_2^2 c_1 f l n^2 + b_2 c_1^2 f l m^2) (72M_{011} + \frac{4}{3}N_{11} + 216M_{111} + 72M_{012} + \frac{52}{15}N_{12} + \frac{6}{5}N_{03} \\
 & \qquad \qquad \qquad - 32M_{022} + 306M_{112} - \frac{32}{5}N_{04} + \frac{27}{5}N_{13} \equiv 0) \\
 & + (b_2^2 c_1 m n^3 + b_2 c_1^2 m^3 n) (81M_{111} - \frac{8}{5}N_{12} + 32M_{022} + 180M_{112} - \frac{14}{3}N_{13} \equiv 0) \\
 & + (b_2^2 f^3 g^2 + c_1^2 f^3 h^2) (\frac{2}{5}N_{12} + \frac{6}{5}N_{03} + 2N_{04} + \frac{6}{5}N_{13} \equiv 0) \\
 & + b_2 c_1 f^3 g h (-48M_{012} - \frac{16}{15}N_{12} - \frac{16}{5}N_{03} - 432M_{004} - 104M_{022} - 48M_{112} \\
 & \qquad \qquad \qquad + 4N_{04} - \frac{16}{5}N_{13} \equiv 0) \\
 & + (a_1 c_1 f^4 m + a_2 b_2 f^4 n) (54M_{011} + 2N_{11} + 81M_{111} + 81M_{012} + 3N_{12} + \frac{33}{10}N_{03} \\
 & \qquad \qquad \qquad + 324M_{004} + 108M_{022} + 108M_{112} + \frac{24}{5}N_{04} + 4N_{13} \equiv 0) \\
 & + b_2 c_1 f^3 l^2 (-96M_{012} - \frac{32}{15}N_{12} - \frac{32}{5}N_{03} - 864M_{004} - 328M_{022} - 96M_{112} - 20N_{04} \\
 & \qquad \qquad \qquad - \frac{32}{5}N_{13} \equiv 0) \\
 & + (b_2^2 f^2 g n^2 + c_1^2 f^2 h m^2) (27M_{011} + N_{11} + 81M_{111} + 36M_{012} + \frac{6}{5}N_{12} + 12M_{022} \\
 & \qquad \qquad \qquad + 108M_{112} \equiv 0)
 \end{aligned}$$

$$\begin{aligned}
& + (b_2 c_1 f^2 g m^2 + b_2 c_1 f^2 h n^2) \left(-36M_{011} - \frac{4}{3}N_{11} - 108M_{111} - 126M_{012} - \frac{22}{5}N_{12} \right. \\
& \quad \left. - \frac{27}{5}N_{03} - 864M_{004} - 304M_{022} - 240M_{112} - 16N_{04} - \frac{48}{5}N_{13} \equiv 0 \right) \\
& + b_2 c_1 f^2 l m n \left(-18M_{011} - \frac{2}{3}N_{11} - 54M_{111} + 72M_{012} - \frac{4}{5}N_{12} + \frac{18}{5}N_{03} + 864M_{004} \right. \\
& \quad \left. + 376M_{022} + 6M_{112} + \frac{128}{5}N_{04} - \frac{3}{5}N_{13} \equiv 0 \right) \\
& + (b_2^2 f n^4 + c_1^2 f m^4) \left(-81M_{111} - \frac{4}{5}N_{12} - 180M_{112} - \frac{14}{5}N_{13} \equiv 0 \right) \\
& + b_2 c_1 f m^2 n^2 \left(-324M_{111} - \frac{8}{5}N_{12} - \frac{12}{5}N_{03} - 48M_{022} - 684M_{112} - \frac{48}{5}N_{04} - \frac{26}{5}N_{13} \equiv 0 \right) \\
& + (a_1 f^5 n + a_2 f^5 m) \left(-54M_{011} - 2N_{11} - 81M_{111} - 81M_{012} - 3N_{12} - \frac{33}{10}N_{03} \right. \\
& \quad \left. - 324M_{004} - 108M_{022} - 108M_{112} - \frac{24}{5}N_{04} - 4N_{13} \equiv 0 \right) \\
& + (b_2 f^4 g l + c_1 f^4 h l) \left(-36M_{012} - \frac{4}{5}N_{12} - \frac{12}{5}N_{03} - 324M_{004} - 108M_{022} - 36M_{112} \right. \\
& \quad \left. - 4N_{04} - \frac{12}{5}N_{13} \equiv 0 \right) \\
& + (b_2 f^3 g m n + c_1 f^3 h m n) \left(-54M_{011} - 2N_{11} - 162M_{111} - 63M_{012} - \frac{19}{5}N_{12} - \frac{27}{10}N_{03} \right. \\
& \quad \left. - 24M_{022} - 216M_{112} - 8N_{04} - \frac{24}{5}N_{13} \equiv 0 \right) \\
& + (b_2 f^3 l n^2 + c_1 f^3 l m^2) \left(-54M_{011} - 2N_{11} - 162M_{111} - 54M_{012} - \frac{18}{5}N_{12} - \frac{3}{5}N_{03} \right. \\
& \quad \left. + 24M_{022} - 216M_{112} + \frac{16}{5}N_{04} - \frac{24}{5}N_{13} \equiv 0 \right) \\
& + (b_2 f^2 m n^3 + c_1 f^2 m^3 n) \left(243M_{111} + \frac{18}{5}N_{12} - 24M_{022} + 504M_{112} + \frac{56}{5}N_{13} \equiv 0 \right) \\
& + f^5 g h \left(18M_{012} + \frac{2}{5}N_{12} + \frac{6}{5}N_{03} + 162M_{004} + 42M_{022} + 18M_{112} - \frac{4}{5}N_{04} + \frac{6}{5}N_{13} \equiv 0 \right) \\
& + f^5 l^2 \left(36M_{012} + \frac{4}{5}N_{12} + \frac{12}{5}N_{03} + 324M_{004} + 120M_{022} + 36M_{112} + \frac{34}{5}N_{04} + \frac{12}{5}N_{13} \equiv 0 \right) \\
& + (f^4 g m^2 + f^4 h n^2) \left(27M_{011} + N_{11} + 81M_{111} + 63M_{012} + \frac{13}{5}N_{12} + \frac{27}{10}N_{03} + 324M_{004} \right. \\
& \quad \left. + 120M_{022} + 144M_{112} + 8N_{04} + \frac{24}{5}N_{13} \equiv 0 \right) \\
& + f^4 l m n \left(54M_{011} + 2N_{11} + 162M_{111} + 18M_{012} + \frac{14}{5}N_{12} - \frac{9}{5}N_{03} - 324M_{004} \right. \\
& \quad \left. - 156M_{022} + 180M_{112} - \frac{64}{5}N_{04} + \frac{12}{5}N_{13} \equiv 0 \right) \\
& + f^3 m^2 n^2 \left(-162M_{111} - \frac{12}{5}N_{12} + \frac{6}{5}N_{03} + 36M_{022} - 324M_{112} + \frac{24}{5}N_{04} - \frac{36}{5}N_{13} \equiv 0 \right)] \\
& + \sum^2 2x_0 x_1 [a_1 b_2^3 c_1^3 (-64M_{011} - 32M_{111} - 32M_{012} \equiv 128) \\
& \quad + a_1 b_2^2 c_1^2 f^2 (144M_{011} + 4N_{11} + 72M_{111} + 72M_{012} + 2N_{12} + \frac{7}{5}N_{03} \equiv -72) \\
& \quad + b_2^3 c_1^2 g m (-4M_{012} + \frac{8}{90}N_{12} \equiv 0) \\
& \quad + b_2^2 c_1^3 h m (96M_{011} + 72M_{111} + 68M_{012} + \frac{8}{90}N_{12} \equiv 0) \\
& \quad + b_2^2 c_1^2 l n (96M_{011} + 72M_{111} + 80M_{012} - \frac{16}{90}N_{12} \equiv 0) \\
& \quad + b_2^3 c_1 f g n (-6M_{012} - \frac{8}{30}N_{12} - \frac{21}{10}N_{03} \equiv 0) \\
& \quad + b_2^2 c_1^2 f h n (-96M_{011} - \frac{8}{3}N_{11} - 72M_{111} - 70M_{012} - \frac{172}{90}N_{12} - \frac{7}{10}N_{03} \equiv 0) \\
& \quad + b_2^2 c_1^2 f l m (-96M_{011} - \frac{8}{3}N_{11} - 72M_{111} - 76M_{012} - \frac{196}{90}N_{12} - \frac{14}{5}N_{03} \equiv 0) \\
& \quad + b_2^3 c_1 n^3 (-36M_{011} - \frac{81}{2}M_{111} - 48M_{012} - \frac{8}{30}N_{12} \equiv 0) \\
& \quad + b_2^2 c_1^2 m^2 n (-72M_{011} + \frac{4}{3}N_{11} - \frac{135}{2}M_{111} - 72M_{012} + \frac{16}{30}N_{12} \equiv 0) \\
& \quad + a_1 b_2 c_1 f^4 (-108M_{011} - 4N_{11} - 54M_{111} - 54M_{012} - 2N_{12} - \frac{7}{5}N_{03} \equiv 0)]
\end{aligned}$$

$$\begin{aligned}
 & + b_2^2 c_1 f^2 g m (9M_{012} + \frac{1}{5}N_{12} + \frac{21}{10}N_{03} = 0) \\
 & + b_2 c_1^2 f^2 h m (-144M_{011} - 4N_{11} - 108M_{111} - 105M_{012} - \frac{92}{30}N_{12} - \frac{21}{10}N_{03} = 0) \\
 & + b_2^2 c_1 f^2 l n (-144M_{011} - 4N_{11} - 108M_{111} - 114M_{012} - \frac{86}{30}N_{12} - \frac{21}{10}N_{03} = 0) \\
 & + b_2^2 c_1 f m n^2 (144M_{011} + \frac{8}{3}N_{11} + 135M_{111} + 156M_{012} + \frac{68}{30}N_{12} + \frac{21}{5}N_{03} = 0) \\
 & + b_2 c_1^2 f^2 m^3 (72M_{011} + \frac{4}{3}N_{11} + \frac{135}{2}M_{111} + 72M_{012} + \frac{52}{30}N_{12} + \frac{14}{5}N_{03} = 0) \\
 & + b_2^2 f^3 g n (\frac{9}{2}M_{012} + \frac{1}{10}N_{12} + \frac{21}{20}N_{03} = 0) \\
 & + b_2 c_1 f^3 h n (144M_{011} + \frac{16}{3}N_{11} + 108M_{111} + \frac{213}{2}M_{012} + \frac{119}{30}N_{12} + \frac{49}{20}N_{03} = 0) \\
 & + b_2 c_1 f^3 l m (144M_{011} + \frac{16}{3}N_{11} + 108M_{111} + 111M_{012} + \frac{122}{30}N_{12} + \frac{35}{10}N_{03} = 0) \\
 & + b_2^2 f^2 n^3 (27M_{011} + N_{11} + \frac{81}{2}M_{111} + 36M_{012} + \frac{14}{10}N_{12} = 0) \\
 & + b_2 c_1 f^2 m^2 n (-54M_{011} - 2N_{11} - \frac{81}{2}M_{111} - 54M_{012} - \frac{18}{10}N_{12} - \frac{21}{5}N_{03} = 0) \\
 & + a_1 f^6 (27M_{011} + N_{11} + \frac{27}{2}M_{111} + \frac{27}{2}M_{012} + \frac{1}{2}N_{12} + \frac{7}{20}N_{03} = 0) \\
 & + b_2 f^4 g m (-\frac{9}{2}M_{012} - \frac{1}{10}N_{12} - \frac{21}{20}N_{03} = 0) \\
 & + c_1 f^4 h m (54M_{011} + 2N_{11} + \frac{81}{2}M_{111} + \frac{81}{2}M_{012} + \frac{15}{10}N_{12} + \frac{21}{20}N_{03} = 0) \\
 & + b_2 f^4 l n (54M_{011} + 2N_{11} + \frac{81}{2}M_{111} + \frac{81}{2}M_{012} + \frac{15}{10}N_{12} + \frac{21}{20}N_{03} = 0) \\
 & + b_2 f^3 m n^2 (-108M_{011} - 4N_{11} - \frac{243}{2}M_{111} - 117M_{012} + \frac{44}{10}N_{12} - \frac{21}{10}N_{03} = 0) \\
 & + c_1 f^3 m^3 (-54M_{011} - 2N_{11} - 54M_{111} - 54M_{012} - 2N_{12} - \frac{7}{5}N_{03} = 0) \\
 & + f^5 h n (-54M_{011} - 2N_{11} - \frac{81}{2}M_{111} - \frac{81}{2}M_{012} - \frac{15}{10}N_{12} - \frac{21}{20}N_{03} = 0) \\
 & + f^5 l m (-54M_{011} - 2N_{11} - \frac{81}{2}M_{111} - \frac{81}{2}M_{012} - \frac{15}{10}N_{12} - \frac{21}{20}N_{03} = 0) \\
 & + f^4 m^2 n (81M_{011} + 3N_{11} + 81M_{111} + 81M_{012} + 3N_{12} + \frac{21}{10}N_{03} = 0)] \\
 & {}_1^2 [b_2^3 c_1^3 h (-64M_{011} = 384) \\
 & + b_2^3 c_1^2 m n (24M_{011} + \frac{8}{9}N_{11} = 0) \\
 & + b_2^2 c_1^2 f^2 h (144M_{011} + 4N_{11} = -216) \\
 & + b_2^3 c_1 f n^2 (36M_{011} + \frac{4}{3}N_{11} = 0) \\
 & + b_2^2 c_1^2 f m^2 (-24M_{011} - \frac{8}{9}N_{11} = 0) \\
 & + b_2^2 c_1 f^2 m n (-90M_{011} - \frac{10}{3}N_{11} = 0) \\
 & + b_2 c_1 f^4 h (-108M_{011} - 4N_{11} = 0) \\
 & + b_2 c_1 f^3 m^2 (54M_{011} + 2N_{11} = 0) \\
 & + b_2^2 f^3 n^2 (-27M_{011} - N_{11} = 0) \\
 & + b_2 f^4 m n (54M_{011} + 2N_{11} = 0) \\
 & + f^6 h (27M_{011} + N_{11} = 0) \\
 & + f^5 m^2 (-27M_{011} - N_{11} = 0)] \\
 & + 2x_1 x_2 [b_2^3 c_1^3 l (-64M_{011} = 384) \\
 & + (b_2^3 c_1^2 n^2 + b_2^3 c_1^3 m^2) (60M_{011} + \frac{4}{9}N_{11} = -288) \\
 & + b_2^2 c_1^2 f^2 l (144M_{011} + 4N_{11} = -216)
 \end{aligned}$$

$$\begin{aligned}
& + b_2^2 c_1^2 f m n (-120 M_{011} - \frac{16}{9} N_{11} \equiv 432) \\
& + (b_2^2 c_1 f^2 n^2 + b_2 c_1^2 f^2 m^2) (-81 M_{011} - 3 N_{11} \equiv 0) \\
& + b_2 c_1 f^4 l (-108 M_{011} - 4 N_{11} \equiv 0) \\
& + b_2 c_1 f^3 m n (162 M_{011} + 6 N_{11} \equiv 0) \\
& + (b_2 f^4 n^2 + c_1 f^4 m^2) (27 M_{011} + N_{11} \equiv 0) \\
& + f^6 l (27 M_{011} + N_{11} \equiv 0) \\
& + f^5 m n (-54 M_{011} - 2 N_{11} \equiv 0)] \\
= & -4 b_2^2 c_1^2 [x_0^2 (-16 a b_2 c_1 + 9 a f^2) + 2 x_0 x_1 (-32 a_1 b_2 c_1 + 18 a_1 f^2) \\
& + 2 x_0 x_2 (-32 a_2 b_2 c_1 + 18 a_2 f^2) + x_1^2 (-96 b_2 c_1 h + 54 f^2 h) \\
& + 2 x_1 x_2 (-96 b_2 c_1 l + 72 b_2 n^2 + c_1 m^2 + 54 f^2 l - 108 f m n) \\
& + x_2^2 (-96 b_2 c_1 g + 54 f^2 g)].
\end{aligned}$$

Therefore the expression for the satellite is correct.

It follows directly from (1'), when $(a\xi)^{12}$ is taken as $s^3 - 27t^2$, which in this case becomes $4b_2^2 c_1^2 (-16b_2 c_1 + 9f^2)$, that

$$T - (\alpha x)^4 (a\xi)^4 = (x\xi)^2 S.$$

Therefore the asymptotic equation of the quartic is

$$\begin{aligned}
& (s^3 - 27t^2) (\alpha x)^4 - 6 (ss_1^2 - 27t_1^2) (x\xi)^2 \\
& + \frac{8}{13} (22s_1^3 - 9ss_1s_2 - 405t_1t_2 + \frac{540}{7}tt_3) (x\xi)^3 \\
& + \frac{9}{13} (-\frac{49}{9}s^2s_4 + \frac{45}{2}ss_2^2 - \frac{21}{2}s_1^2s_2 - \frac{875}{7}tt_4 + \frac{1485}{7}t_1t_3) (x\xi)^4 = 0.
\end{aligned}$$

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